Characterizing chainable and tree-like continua

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Chainable continua

Continuum = compact connected Hausdorff space

Definition 1

An open cover $\mathcal{U}$ of X is called *chain-open cover* of X if for $\mathcal{U}$ there is an enumeration $\mathcal{U} = \{U_1, \dots, U_m\}$ such that

$$U_i \cap U_j \neq \emptyset \iff |i - j| \leqslant 1 \text{ for all } 1 \leqslant i, j \leqslant n.$$

Definition 2

A continuum X is called *chainable* if each open cover of X there is a chain-open cover refinement.

Remark

In metric case a continuum X is chainable iff for each $\varepsilon > 0$ there is a chain-open cover $\mathcal{U}$ of X with mesh$(\mathcal{U}) < \varepsilon$.

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Definition 3

A continuum X is called *n-chainable*, where $n \geqslant 3$, if for any open cover $\mathcal{U}$ with $|\mathcal{U}| \leqslant n$ there is an open-chain refinement.

A continuum X is a chainable iff X is n-chainable for each n.

Remark

If X is n-chainable, then X is m-chainable for each $m \leqslant n$.

In the paper T. Banakh, P. Bankston, B. Raines, W. Ruitenburg, *Chainability and Hemmingsen's theorem*, Topology Appl. **153** (2006) 2462–2468, it was announced (without proof) that 4-chainable implies n-chainability for each n. In other words 4-chainabilty implies chainability.

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Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be a finite family of subsets of an arbitrary space X and let $p_1, \dots, p_n$ ba a system of points of an euclidean space $\mathbb{R}^m$. A nerve of the family $\mathcal{U}$ is the simplicial complex $\mathcal{N}(U)$ formed by simplexes $\langle p_{i_0}, \dots, p_{i_l}\rangle$ such that $U_{i_0} \cap \dots \cap U_{i_l} \neq \emptyset$.

Theorem 1

If X is a normal space and $\mathcal{U} = \{U_1, \dots, U_n\}$ is a finite open cover of X then there is a map from X into the nerve $\mathcal{N}(\mathcal{U})$ i.e. there is a continuous map $\kappa\colon X \to |\mathcal{N}(U)|$ such that

$$\kappa^{-1}(\mathrm{st}_{|\mathcal{N}(\mathcal{U})|} p_i) \subseteq U_i$$

for every $i = 1, \dots, n$.

$|\mathcal{N}(U)| = \bigcup \mathcal{N}(\mathcal{U})$ – a carrier of the nerve $\mathcal{N}(\mathcal{U})$;
$\mathrm{st}_{|\mathcal{N}(\mathcal{U})|} p_i = |\mathcal{N}(\mathcal{U})| \setminus \cup\{S \in \mathcal{N}(\mathcal{U})\colon p_i \notin S\}$ – a star of vertex p_i in $|\mathcal{N}(\mathcal{U})|$.

$\mathcal{U}$-maps

Definition 4

Let $\mathcal{U}$ be an open cover of a topological space X. A map $f\colon X \to Y$ into a space Y is called a *$\mathcal{U}$-map* if there is an open cover $\mathcal{V}$ of Y whose preimage $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$ refines $\mathcal{U}$.

Remark

Let X and Y be a compact Hausdorff spaces and let $\mathcal{U}$ be an open cover of X. A map $f\colon X \to Y$ is $\mathcal{U}$-map iff for each $y \in Y$ there is $U \in \mathcal{U}$ such that $f^{-1}(y) \subseteq U$.

Theorem 2

If X is a normal space then $\dim X = 1$ iff for any open cover $\mathcal{U}$ of X there is a $\mathcal{U}$-map $f\colon X \to \Gamma$ onto a graph Γ.

Graph = carrier of 1-dimensional simplicial complex.

If $\mathcal{U}$ is a chain then the carrier $|\mathcal{N}(U)|$ of the nerve $\mathcal{N}(\mathcal{U})$ is an arc, so:

Theorem 3

1. *A continuum X is chainable if and only if for any (finite) open cover $\mathcal{U}$ of X there is a $\mathcal{U}$-map from X onto an arc.*
2. *A continuum X is n-chainable iff for each open cover $\mathcal{U}$ of X such that $|\mathcal{U}| \leqslant n$ there is a $\mathcal{U}$-map f from X onto an arc.*

Tree-like continua

A continuum X is said to be tree-like provided that every open cover of X can be refined by a finite open cover having nerve a tree, that is, having nerve a connected acyclic graph. Similarly, to the n-chainability we define the notion of a n-tree-likeness.

A counterpart of the Theorem 3 for the class of tree-like continua is the following:

Theorem 4

1. *A continuum X is a tree-like if and only if for each open cover $\mathcal{U}$ of X there is a tree T and $\mathcal{U}$-map $f\colon X \to T$.*
2. *A continuum X is a n-tree-like if and only if for each open cover $\mathcal{U}$ of X such that $|\mathcal{U}| \leqslant n$ there is a tree T and $\mathcal{U}$-map $f\colon X \to T$.*

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Chainable and tree-like continua are particular cases of the concept of a $\mathfrak{T}$-like continuum. We shall say that a continuum X is *$\mathfrak{T}$-like*, where $\mathfrak{T}$ is a class of continua, if for any open cover $\mathcal{U}$ of X there is a $\mathcal{U}$-map $f\colon X \to T$ onto some space $T \in \mathfrak{T}$.

A continuum X is called *n-$\mathfrak{T}$-like* is for any open cover $\mathcal{U}$ with $|\mathcal{U}| \leqslant n$ there is an $\mathcal{U}$-map $f\colon X \to T \in \mathfrak{T}$.

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Main Theorem

Theorem 5

For a subclass $\mathfrak{T}$ of the class of tree-like continua and a continuum X the following conditions are equivalent:

1. *X is $\mathfrak{T}$-like (i.e. for each open cover $\mathcal{U}$ of X there is a $\mathcal{U}$-map $f\colon X \to T$ of X onto a space $T \in \mathfrak{T}$);*
2. *X is 4-$\mathfrak{T}$-like (i.e for each 4-set open cover $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$ of X there is a $\mathcal{U}$-map $f\colon X \to T$ of X onto a space $T \in \mathfrak{T}$).*

Corollary 1

A continuum X is chainable (resp. tree-like) if and only if each 4-set open cover $\mathcal{U} = \{U_1, U_2, U_3, U_4\}$ of X has a chain-open (resp. tree-open) refinement.

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Theorem 6 (Hemmingsen)

For a compact Hausdorff space X the following conditions are equivalent:

1. $\dim X \leqslant 1$, *which means that any open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ of order $\leqslant 2$;*
2. *each 3-set open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ of order $\leqslant 2$;*
3. *each 3-set open cover $\mathcal{U} = \{U_1, U_2, U_3\}$ of X has an open 3-set refinement $\mathcal{V} = \{V_1, V_2, V_3\}$ with $V_1 \cap V_2 \cap V_3 = \emptyset$;*

Corollary 2

Let $\mathfrak{T}$ be a some class of tree-like continua. If X is a n-$\mathfrak{T}$-like continuum, $n \geqslant 3$, then $\dim X = 1$.

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Lemma 1

For any open cover $\mathcal{U}$ of a topological graph Γ there is a $\mathcal{U}$-map $f : \Gamma \to G$ onto a topological graph of degree $\leqslant 3$.

This lemma can be easily proved by induction with respect to number of branching point of Γ. The following drawing illustrates how to decrease a degree of a selected vertex of a graph.

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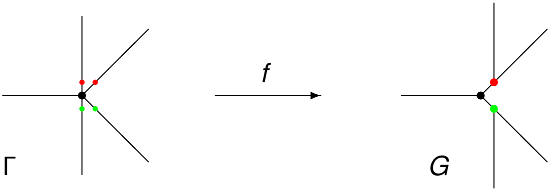

In the next lemma graph G is consider as a combinatorial object.

Lemma 2

Let $G = (V, E)$ be a connected graph with $\deg(G) \leqslant 3$ such that $d(u, v) \geqslant 6$ for any two vertices $u, v \in V$ of order 3. Then there is a 4-coloring $\chi \colon V \to \{1, 2, 3, 4\}$ such that no distinct vertices $u, v \in V$ with $d(u, v) \leqslant 2$ have the same color.

Proof.

Let $V_3 = \{v \in V : \deg(v) = 3\}$ and let $B(v) = \{u \in V \colon \{u, v\} \in E\}$ for each $v \in V$. Since $\deg(G) \leqslant 3$ then $|B(v)| \leqslant 4$ for each $v \in V$. Moreover:

$$v, w \in V_3, v \neq w \Rightarrow B(v) \cap B(w) = \emptyset.$$

So we can define a 4-coloring χ on the union $\bigcup_{v \in V_3} B(v)$ so that χ is injective on each $B(v)$ and $\chi(v) = \chi(w)$ for each $v, w \in V_3$. Next, it remains to color the remaining vertices all of order $\leqslant 2$ by four colors $\chi(x) \neq \chi(y)$ if $d(x, y) \leqslant 2$. It is easy to check that this always can be done. □

Take a class $\mathfrak{T}$ of tree-like continua and assume that X is a continuum such that for any 4-set open cover $\mathcal{U}_4$ of X there is a $\mathcal{U}_4$-map $f\colon X \to T$ onto a space $T \in \mathfrak{T}$. We should prove that such a map exists for any (finite) open cover $\mathcal{U}$ of X.

By the corollary 2, there is a $\mathcal{U}$-map $f : X \to \Gamma$ onto a topological graph Γ. Because of Lemma 1, we can assume that $\deg(\Gamma) \leqslant 3$. Selecting vertices on edges of Γ, we find so fine triangulation $G = (V, E)$ of the topological graph Γ that

- the distance between any vertices of order 3 in the path metric of G is $\geqslant 6$;
- the cover $\{f^{-1}(\mathrm{st}_\Gamma(v)) : v \in V\}$ of X is inscribed into $\mathcal{U}$.

Lemma 2 implies that there is a 4-coloring $\chi : V \to \{1, 2, 3, 4\}$ of V such that no vertices $u, v \in V$ with $0 < d(u, v) \leqslant 2$ are monochromatic.

For $i \in \{1, 2, 3, 4\}$ consider the open set

$$V_i = \cup\{\mathrm{st}_\Gamma v\colon \chi(v) = i\}.$$

The sets $V_1, \dots, V_4$ cover Γ. Then for the 4-set cover $\mathcal{V} = \{f^{-1}(V_i)\colon i \leqslant 4\}$ of X we can find a $\mathcal{V}$-map $g\colon X \to Y$ to a continuum $Y \in \mathfrak{T}$.
Since Y is tree-like, there is a map $\pi : Y \to T$ onto a topological tree such that the composition $h = \pi \circ g : X \to T$ still is a $\mathcal{V}$-map. Take a triangulation $H = (V_T, E_T)$ of T so fine that the cover $\{h^{-1}(\mathrm{st}_T t)\colon t \in V_T\}$ is inscribed into the cover $\mathcal{V}$. Consequently, for each vertex $t \in V_T$ we can find a number $\xi(t) \in \{1, 2, 3, 4\}$ such that

$$h^{-1}(\mathrm{st}_T t) \subseteq f^{-1}(V_{\xi(t)}).$$

Using the property of the coloring χ we can prove that

$$(\forall t \in V_T)(\exists! v_t \in V)(\xi(t) = \chi(v_t)).$$

So,

$$g^{-1}(\pi^{-1}(\mathrm{st}_T t)) = h^{-1}(\mathrm{st}_T t) \subseteq f^{-1}(\mathrm{st}_\Gamma v_t) \subseteq U$$

for some $U \in \mathcal{U}$. It means that g is $\mathcal{U}$-map, so this finishes the proof.

Theorem 7

For a 1-dimensional continuum X the following conditions are equivalent:

1. *each map from X into the circle is homotopic to a constant map;*
2. *X is 3-chainable;*
3. *X is 3-tree-like.*

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There is a 1-dimensional continuum for which each map from X into the circle is homotopic to a constant map but it is not tree-like – J. H. Case, R. E. Chamberlin *Characterizations of tree-like continua*, Pacific J. Math. **10** (1960) 73–84.